

# INVERSION OF SERIES AND THE COHOMOLOGY OF THE MODULI SPACES $\mathcal{M}_{0,n}^\delta$ .

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ABSTRACT. For  $n \geq 3$ , let  $\mathcal{M}_{0,n}$  denote the moduli space of genus 0 curves with  $n$  marked points, and  $\overline{\mathcal{M}}_{0,n}$  its smooth compactification. A theorem due to Ginzburg, Kapranov and Getzler states that the inverse of the exponential generating series for the Poincaré polynomial of  $H^\bullet(\mathcal{M}_{0,n})$  is given by the corresponding series for  $H^\bullet(\overline{\mathcal{M}}_{0,n})$ . In this paper, we prove that the inverse of the ordinary generating series for the Poincaré polynomial of  $H^\bullet(\mathcal{M}_{0,n})$  is given by the corresponding series for  $H^\bullet(\mathcal{M}_{0,n}^\delta)$ , where  $\mathcal{M}_{0,n} \subset \mathcal{M}_{0,n}^\delta \subset \overline{\mathcal{M}}_{0,n}$  is a certain smooth affine scheme.

## 1. INTRODUCTION

For  $n \geq 3$ , let  $\mathcal{M}_{0,n}$  be the moduli space, defined over  $\mathbf{Z}$ , of smooth  $n$ -pointed curves of genus zero, and let  $\mathcal{M}_{0,n} \subset \overline{\mathcal{M}}_{0,n}$  denote its smooth compactification, due to Deligne-Mumford and Knudsen. In [1], an intermediary space  $\mathcal{M}_{0,n}^\delta$ , which satisfies

$$\mathcal{M}_{0,n} \subset \mathcal{M}_{0,n}^\delta \subset \overline{\mathcal{M}}_{0,n},$$

was defined in terms of explicit polynomial equations. It is a smooth affine scheme over  $\mathbf{Z}$ . The automorphism group of  $\mathcal{M}_{0,n}$  is the symmetric group  $\mathfrak{S}_n$  permuting the  $n$  marked points, and this gives rise to a decomposition (see [1]),

$$\overline{\mathcal{M}}_{0,n} = \bigcup_{\sigma \in \mathfrak{S}_n} \sigma(\mathcal{M}_{0,n}^\delta).$$

Thus  $\mathcal{M}_{0,n}^\delta$  defines a symmetric set of canonical affine charts for  $\overline{\mathcal{M}}_{0,n}$ .

In this note, we compute the dimensions

$$a_{n,i} := \dim_{\mathbf{Q}} H^i(\mathcal{M}_{0,n}^\delta; \mathbf{Q})$$

of the de Rham cohomology of  $\mathcal{M}_{0,n}^\delta$  for all  $i$  and  $n$ . Our main result can be expressed in terms of generating series, as follows. If  $X$  is a smooth scheme over  $\mathbf{Q}$  of dimension  $d$ , we will denote its Euler characteristic (or rather, Poincaré polynomial) by:

$$e(X)(q) = \sum_i (-1)^i \dim_{\mathbf{Q}} H^i(X; \mathbf{Q}) q^{d-i}.$$

Consider the exponential generating series:

$$\begin{aligned} g(x) &:= x - \sum_{n=2}^{\infty} e(\mathcal{M}_{0,n+1})(q^2) \frac{x^n}{n!}, \\ \overline{g}(x) &:= x + \sum_{n=2}^{\infty} e(\overline{\mathcal{M}}_{0,n+1})(q) \frac{x^n}{n!}. \end{aligned}$$

The following formula is due to Ginzburg-Kapranov ([4], theorem 3.3.2) and Getzler ([3], §5.8)

$$(1) \quad \overline{g}(g(x)) = g(\overline{g}(x)) = x.$$

In this paper, we will consider the ordinary generating series:

$$\begin{aligned} f(x) &:= x - \sum_{n=2}^{\infty} e(\mathcal{M}_{0,n+1})(q) x^n, \\ f_{\delta}(x) &:= x + \sum_{n=2}^{\infty} e(\mathcal{M}_{0,n+1}^{\delta})(q) x^n. \end{aligned}$$

**Theorem 1.1.** *The following inversion formula holds:*

$$(2) \quad f(f_{\delta}(x)) = f_{\delta}(f(x)) = x.$$

Using the well-known formula

$$(3) \quad e(\mathcal{M}_{0,n+1})(q) = \prod_{i=2}^{n-1} (q - i)$$

and the purity of  $\mathcal{M}_{0,n+1}^{\delta}$ , we deduce a recurrence relation for  $e(\mathcal{M}_{0,n+1}^{\delta})$  from (2), and hence also for the Betti numbers  $a_{n,i}$ . The proof of equation (2) uses the fact that the coefficients in the Lagrange inversion formula are precisely given by the combinatorics of Stasheff polytopes, which in turn determine the structure of the mixed Tate motive underlying  $\mathcal{M}_{0,n}^{\delta}$ .

In the special case  $q = 0$ , the series  $f(x)$  reduces to  $x - \sum_{n=2}^{\infty} (n-1)! x^n$ , which is essentially the generating series for the operad  $\mathfrak{L}\mathfrak{ie}$ . Comparing equation (2) to Lemma 8 in [10] we find that the dimensions  $a_{n,n-3} = H^{n-3}(\mathcal{M}_{0,n}^{\delta}; \mathbf{Q})$  are precisely the numbers of prime generators for  $\mathfrak{L}\mathfrak{ie}$ . We expect that there should be an explicit bijection between  $H^{n-3}(\mathcal{M}_{0,n}^{\delta}; \mathbf{Q})$  and the set of prime generators described in the proof of Proposition 6 in [10], and, more generally, an operad-theoretic interpretation of equation (2) for all  $q$ .

**Remark 1.2.** The numbers  $a_{n,n-3}$  count the number of convergent period integrals on the moduli space  $\mathcal{M}_{0,n}$  defined in [2], called ‘cell-zeta values’. Specifically, there is a connected component  $X_{\delta}$  of the set of real points  $\mathcal{M}_{0,n}(\mathbf{R}) \subset \mathcal{M}_{0,n}^{\delta}(\mathbf{R})$  whose closure  $\overline{X}_{\delta} \subset \mathcal{M}_{0,n}^{\delta}(\mathbf{R})$  is a compact manifold with corners, and is combinatorially a Stasheff polytope. For any  $\omega \in H^{n-3}(\mathcal{M}_{0,n}^{\delta}; \mathbf{Q})$ , one can consider the integral

$$I(\omega) = \int_{X_{\delta}} \omega \in \mathbf{R},$$

which is the period of a framed mixed Tate motive over  $\mathbf{Z}$ , see [6]. By a theorem in [1], the number  $I(\omega)$  is a  $\mathbf{Q}$ -linear combination of multiple zeta values. For example, when  $n = 5$ , we have  $a_{5,2} = 1$ , and there is essentially a unique such integral. Identifying  $\mathcal{M}_{0,5}$  with  $\{(t_1, t_2) \in (\mathbb{P}^1 \setminus \{0, 1, \infty\})^2, t_1 \neq t_2\}$ , we can write  $I(\omega)$  as

$$\int_{0 < t_1 < t_2 < 1} \frac{dt_1 dt_2}{(1 - t_1)t_2} = \zeta(2).$$

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## 2. GEOMETRY OF $\mathcal{M}_{0,n}^{\delta}$

We recall some geometric properties of  $\mathcal{M}_{0,n}^{\delta}$  from [1]. The set of real points  $\mathcal{M}_{0,n}(\mathbf{R})$  is not connected but has  $n!/2n$  components, and they can be indexed by the set of dihedral structures<sup>1</sup>  $\delta$  on the set  $\{1, \dots, n\}$ . Let  $X_{\delta}$  denote one such

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<sup>1</sup>A dihedral structure on a set  $S$  is an identification of the elements of  $S$  with the edges of an unoriented polygon, i.e., considered modulo dihedral symmetries.

connected component. Its closure in the real moduli space

$$\overline{X}_\delta \subset \overline{\mathcal{M}}_{0,n}(\mathbf{R})$$

is a compact manifold with corners. The variety  $\mathcal{M}_{0,n}^\delta \subset \overline{\mathcal{M}}_{0,n}$  is then defined to be the complement  $\overline{\mathcal{M}}_{0,n} \setminus A_\delta$ , where  $A_\delta$  is the set of all irreducible divisors  $D \subset \overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$  which do not meet the closed cell  $\overline{X}_\delta$ . Conversely, every irreducible divisor  $D \subset \overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$  which does meet the closed cell  $\overline{X}_\delta$ , defines an irreducible divisor  $D \cap \mathcal{M}_{0,n}^\delta \subset \mathcal{M}_{0,n}^\delta \setminus \mathcal{M}_{0,n}$ . In the case  $n = 4$ , we have:

$$\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\} , \quad \mathcal{M}_{0,4}^\delta \cong \mathbb{P}^1 \setminus \{\infty\} , \quad \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1 ,$$

where  $X_\delta$  is the open interval  $(0, 1)$  and  $\overline{X}_\delta$  is the closed interval  $[0, 1]$ .

In the case  $n = 5$ , one can take four points in general position in  $\mathbb{P}^2$  and identify  $\mathcal{M}_{0,5}$  with the complement of a configuration of six lines passing through each pair of points. The compactification  $\overline{\mathcal{M}}_{0,5}$  is obtained by blowing up these four points, giving a total of ten boundary divisors. Picturing  $\mathbb{P}^2$  minus the six lines one sees that the set of real points  $\mathcal{M}_{0,5}(\mathbf{R})$  has exactly 12 connected components which are triangles. Choosing one of these components  $X_\delta$ , and blowing up only the two points which meet  $X_\delta$  yields a space in which the boundary divisors incident to  $X_\delta$  form a pentagon. The space  $\mathcal{M}_{0,5}^\delta$  is obtained by removing all divisors of  $\overline{\mathcal{M}}_{0,5}$  except the pentagon which bounds  $\overline{X}_\delta$ . Thus we obtain twelve isomorphic varieties  $\mathcal{M}_{0,5}^\delta$ , one for each connected component of  $\mathcal{M}_{0,5}(\mathbf{R})$ .

In general,  $\overline{X}_\delta \subseteq \mathcal{M}_{0,n}^\delta(\mathbf{R})$  has the combinatorial structure of a Stasheff polytope. Its faces of codimension  $k$  are in bijection with the set of decompositions of a regular  $n$ -gon into  $k + 1$  polygons (with at least 3 sides) by  $k$  non-intersecting chords. Suppose, for each  $i \geq 1$ , that there are  $\lambda_i(D)$  polygons in a decomposition  $D$  which has  $i + 2$  sides. Then the corresponding face is

$$F_D \cong \prod_{i=1}^{n-2} \prod_{j=1}^{\lambda_i(D)} \overline{X}_{i+2} ,$$

and  $\overline{X}_{i+2}$  has itself the combinatorial structure of a Stasheff  $i$ -polytope. Note that  $\overline{X}_3$  and  $\mathcal{M}_{0,3}$  are just points. Since a closed polytope is the disjoint union of its open faces, we deduce the following stratification for  $\mathcal{M}_{0,n}^\delta$ :

$$(4) \quad \mathcal{M}_{0,n}^\delta = \coprod_D i_D \left( \prod_{i=1}^{n-2} \prod_{j=1}^{\lambda_i(D)} \mathcal{M}_{0,i+2} \right) .$$

Here, the disjoint union is taken over all decompositions  $D$  of a regular  $n$ -gon, and  $i_D$  is the isomorphism which restricts to the inclusion of each face  $F_D \hookrightarrow \overline{X}_\delta$ . The empty dissection corresponds to the inclusion of the open stratum  $\mathcal{M}_{0,n}$ .

**Example 2.1.** There are nine chords in a regular hexagon, six of which decompose it into a pentagon and trigon, and three of which decompose it into two tetragons. It then has 21 decompositions into three pieces (a tetragon and two triangles), and 14 into four triangles. Therefore equation (4) can be abbreviated:

$$(5) \quad \mathcal{M}_{0,6}^\delta = \mathcal{M}_{0,6} \cup \left( 6 \mathcal{M}_{0,5} \cup 3 \mathcal{M}_{0,4}^2 \right) \cup 21 \mathcal{M}_{0,4} \cup 14 \mathcal{M}_{0,3} .$$

### 3. PURITY

Since  $\mathcal{M}_{0,n}^\delta$  is stratified by products of varieties  $\mathcal{M}_{0,r}$ , which are isomorphic to an affine complement of hyperplanes and therefore of Tate type, it follows that  $H^i(\mathcal{M}_{0,n}^\delta)$  defines an element in the category of mixed Tate motives over  $\mathbf{Q}$ . In fact, it was proved in [1] that  $\mathcal{M}_{0,n}^\delta$  is smooth and affine, so it follows by a theorem due to Grothendieck that its cohomology is generated by global regular forms.

Using the well-known fact that  $H^i(\mathcal{M}_{0,n})$  is pure [3], it follows that the subspace  $H^i(\mathcal{M}_{0,n}^\delta)$  is also pure. We can therefore work inside the semisimple subcategory (or Grothendieck group) generated by pure Tate motives. We have that,

$$(6) \quad H^i(\mathcal{M}_{0,n}^\delta) \cong \mathbf{Q}(-i)^{a_{n,i}}.$$

The purity of the spaces  $\mathcal{M}_{0,n}^\delta$  has the important consequence that we have an equality of Poincaré polynomials (i.e. not only of Euler characteristics),

$$(7) \quad e(\mathcal{M}_{0,n}^\delta) = \sum_D \left( \prod_{i=1}^{n-2} \prod_{j=1}^{\lambda_i(D)} e(\mathcal{M}_{0,i+2}) \right).$$

#### 4. DECOMPOSITIONS OF REGULAR $n$ -GONS

If  $\lambda$  is a partition of a number, we define  $\lambda_i$  to be the number of times  $i$  appears in this partition. For each partition  $\lambda$  of  $n-2$ , we then define  $P(\lambda)$  to be the number of choices of  $-1 + \sum_i \lambda_i$  non-intersecting chords of an  $n$ -regular polygon that gives rise, for each  $i$ , to  $\lambda_i$  subpolygons with  $i+2$  sides. Thus,  $P(\lambda)$  counts the number of decompositions of an  $n$ -gon of given combinatorial type. This number is found to be equal to (see Ex. 2.7.14 on p. 127 in [5]):

$$(8) \quad P(\lambda) = \frac{(n-2 + \sum_i \lambda_i)!}{(n-1)! \prod_i (\lambda_i!)}.$$

Combining this result and (7) we find that,

$$(9) \quad e(\mathcal{M}_{0,n}^\delta) = \sum_{\lambda \vdash n-2} P(\lambda) \cdot \prod_{i=1}^{n-2} e(\mathcal{M}_{0,i+2})^{\lambda_i}.$$

Using equation (3) we can now compute the  $a_{n,i}$ 's for any  $i$  and  $n$ ,

**Example 4.1.** From Example (5), we have

$$e(\mathcal{M}_{0,6}^\delta) = (q-2)(q-3)(q-4) + 6(q-2)(q-3) + 3(q-2)^2 + 21(q-2) + 14,$$

which reduces to  $q^3 + 5q - 4$ . In particular,  $a_{6,3} = \dim_{\mathbf{Q}} H^3(\mathcal{M}_{0,6}^\delta, \mathbf{Q}) = 4$ .

Clearly  $a_{n,0} = 1$  for all  $n$ , and it is also easy to see that  $a_{n,1} = 0$  for all  $n$ . In the following table we present the results for  $n$  from five to eleven.

	$a_{n,1}$	$a_{n,2}$	$a_{n,3}$	$a_{n,4}$	$a_{n,5}$	$a_{n,6}$	$a_{n,7}$	$a_{n,8}$
$\mathcal{M}_{0,5}^\delta$	0	1						
$\mathcal{M}_{0,6}^\delta$	0	5	4					
$\mathcal{M}_{0,7}^\delta$	0	15	28	22				
$\mathcal{M}_{0,8}^\delta$	0	35	112	206	144			
$\mathcal{M}_{0,9}^\delta$	0	70	336	1063	1704	1089		
$\mathcal{M}_{0,10}^\delta$	0	126	840	3999	10848	15709	9308	
$\mathcal{M}_{0,11}^\delta$	0	210	1848	12255	49368	119857	159412	88562

There are no entries above the diagonal, because  $\mathcal{M}_{0,n}^\delta$  is affine. For small  $i$ , one can use (9) to write down explicit formulae for  $a_{n,i}$  as a function of  $n$ , e.g.,

$$a_{n,2} = \binom{n-1}{4} \quad \text{and} \quad a_{n,3} = 4 \binom{n}{6}.$$

Finally, setting  $q = 0$  in (9) gives the following closed formula for the dimension  $a_{n,l}$  of the middle-dimensional de Rham cohomology of  $\mathcal{M}_{0,n}^\delta$ , where  $l := n - 3$ ,

$$(10) \quad \dim_{\mathbf{Q}} H^l(\mathcal{M}_{0,n}^\delta; \mathbf{Q}) = \sum_{\lambda \vdash l+1} P(\lambda) \cdot \prod_{i=1}^{l+1} \left( (-1)^l (l+1)! \right)^{\lambda_i}.$$

## 5. AN INVERSION FORMULA

*Proof of theorem 1.1.* The proof is immediate on comparing equation (9) with the combinatorial interpretation of Lagrange's formula for the inversion of series in one variable (see [9], equation (4.5.12), p. 412). More precisely, consider the formal power series:

$$u(x) = x - \sum_{i=2}^{\infty} u_i x^i$$

Lagrange's formula states that the formal solution to  $v(u(x)) = x$  is given by

$$v(x) = x + \sum_{i=2}^{\infty} v_i x^i$$

where  $v_2 = u_2$ ,  $v_3 = 2u_2^2 + u_3$ ,  $v_4 = 5u_2^3 + 5u_2u_3 + u_4$ , and in general:

$$v_n = \sum_{\lambda \vdash n-1} P(\lambda) \cdot \prod_{i=1}^{n-1} u_i^{\lambda_i}, \quad \text{for } n \geq 2.$$

The theorem follows from (9) on setting  $u_i = e(\mathcal{M}_{0,i+1})$ .  $\square$

**Remark 5.1.** There is a stratification of  $\overline{\mathcal{M}}_{0,n}$  similar to the one described by (4) for  $\mathcal{M}_{0,n}^\delta$ , but where  $P(\lambda)$  should be replaced by  $T(\lambda)$ , and where  $T(\lambda)$  is the number of dual graphs of  $n$ -pointed stable curves of genus zero that has  $\lambda_i$  components with a sum of  $i+2$  marked points and nodes. Now note that from the proof of theorem 1.1 and (1) it follows that  $T(\lambda) = P(\lambda) \cdot (n-1)! / \prod_i (i+1)!^{\lambda_i}$ .

In the special case when  $q = 0$ , we deduce the following corollary.

**Corollary 5.2.** *The generating series for the dimensions  $\dim_{\mathbf{Q}} H^{n-3}(\mathcal{M}_{0,n}^\delta; \mathbf{Q})$  is obtained by inverting the series*

$$\sum_{n=1}^{\infty} (n-1)! x^n = x + x^2 + 2x^3 + 6x^4 + \dots$$

**Remark 5.3.** The cohomology of  $\mathcal{M}_{0,n}$  is a module over the symmetric group  $\mathfrak{S}_n$  with  $n$  elements, whose representation theory can for instance be found in [3] or [8]. The dihedral subgroup  $D_{2n}$  which stabilizes a dihedral ordering  $\delta$  acts upon the affine space  $\mathcal{M}_{0,n}^\delta$ , and hence its cohomology. It therefore would be interesting to compute the character of this group action on  $H^\bullet(\mathcal{M}_{0,n}^\delta)$ , and compare its equivariant generating series to the one obtained by restriction  $\text{Res}_{D_{2n}}^{\mathfrak{S}_n} H^\bullet(\mathcal{M}_{0,n})$ .

## 6. A RECURRENCE RELATION

Let us alter our series slightly and put  $F(x) := -f(-x)$  and  $F_\delta(x) := -f_\delta(-x)$ . By theorem 1.1 we find that  $F_\delta(F(x)) = F(F_\delta(x)) = x$ . The series  $F(x)$  is easily seen to satisfy the differential equation:

$$x^2 F'(x) = (F(x) - x)(xq + 1).$$

By differentiating  $F_\delta(F(x)) = x$ , we have  $F'_\delta(F(x))F'(x) = 1$ . Substituting the previous expression for  $F'(x)$  gives:

$$F'_\delta(F(x))(F(x) - x)(xq + 1) = x^2.$$

By changing variables  $y = F(x)$ , where  $F_\delta(y) = F_\delta(F(x)) = x$ , we obtain:

$$F'_\delta(y)(y - F_\delta(y))(q F_\delta(y) + 1) = F_\delta(y)^2 .$$

Expanding out gives:

$$yF'_\delta - F_\delta F'_\delta - F_\delta^2 + qyF_\delta F'_\delta - qF_\delta^2 F'_\delta = 0 .$$

If we write

$$F_\delta(y) = \sum_{n=1}^{\infty} a_n y^n ,$$

then the coefficient of  $y^n$  is exactly:

$$na_n - \sum_{k+l=n+1} ka_k a_l - \sum_{k+l=n} a_k a_l + q \sum_{k+l=n} ka_k a_l - q \sum_{k+l+m=n+1} ka_k a_l a_m = 0 .$$

Decomposing the first sum  $\sum_{k+l=n+1} ka_k a_l = (n+1)a_1 a_n + \sum_{k=2}^{n-1} ka_k a_{n+1-k}$ , and using the fact that  $a_1 = 1$ , gives the recurrence relation:

$$a_n = - \sum_{\substack{k+l=n+1, \\ k, l \geq 2}} ka_k a_l + \sum_{k+l=n} (qk-1)a_k a_l - q \sum_{k+l+m=n+1} ka_k a_l a_m .$$

**Theorem 6.1.** *The recurrence relation*

$$a_n = - \sum_{\substack{k+l=n+1, \\ k, l \geq 2}} ka_k a_l + \sum_{k+l=n} (qk-1)a_k a_l - q \sum_{k+l+m=n+1} ka_k a_l a_m ,$$

with initial conditions  $a_0 = 0$ ,  $a_1 = 1$ , has a unique solution given by

$$a_n = (-1)^{n+1} e(\mathcal{M}_{0,n+1}^\delta) .$$

In the special case  $q = 0$ , we have the following corollary. Note that in theorem 9 of [10] there is an equivalent presentation of this recurrence relation.

**Corollary 6.2.** *The dimensions  $b_n := \dim_{\mathbf{Q}} H^{n-2}(\mathcal{M}_{0,n+1}^\delta; \mathbf{Q})$  are the unique solutions to the recurrence relation:*

$$b_n = \sum_{\substack{k+l=n+1, \\ k, l \geq 2}} kb_k b_l + \sum_{k+l=n} b_k b_l , \text{ for } n \geq 2 ,$$

with initial conditions  $b_0 = 0, b_1 = -1$ .

*Proof.* Set  $b_n = -a_n|_{q=0}$  in the previous theorem.  $\square$

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